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**LETTER TO THE EDITOR**

# Derivation properties of a deformed Poisson algebra and the quantisation problem

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**Abstract.** We examine the derivation properties of a deformed Poisson algebra of classical observables. Our considerations point to possibly new realisations concerning the quantisation mapping.

The mathematical aspects of the quantisation problem have attracted a great deal of attention in recent years (Groenwold 1946, Van Hove 1951, Joseph 1970, Chernoff 1969, Abraham and Marsden 1978, Segal 1960, Auslander and Kostant 1966, Kostant 1967/8 (cf Kostant 1970), Souriau 1970, Kirillov 1976, Bayen *et al* 1978). In this letter we will attempt to make a connection between two different approaches to this problem. Our hope is that the realisations emerging from the proposed synthesis will contribute toward a better understanding of the quantisation process.

Throughout this letter P denotes the set of all complex polynomials in the $2n$ real variables $p_i$ and $q_i$. As is well known, P acquires a Lie algebra structure via the Poisson bracket

$$\{f(p,q), g(p,q)\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right). \tag{1}$$

By $\mathscr{A}$ we denote the associative, distributive algebra over $\mathbb{C}$ generated by finite linear combinations and finite powers of the elements $\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n$ and $\mathbb{1}$. $\mathscr{A}$ becomes a Lie algebra via the usual Heisenberg relations

$$\hat{q}_i\hat{q}_j - \hat{q}_j\hat{q}_i = \hat{p}_i\hat{p}_j - \hat{p}_j\hat{p}_i = 0, \qquad \hat{q}_i\hat{p}_j - \hat{p}_j\hat{q}_i = \delta_{ij}, \tag{2}$$

and as such it will be denoted by $\mathscr{P}$.

Joseph (1970) has noted the lack of isomorphism between the Lie algebras P and $\mathscr{P}$. This is because $\mathscr{P}$ possesses only inner derivations while P possesses, according to Wollenberg's (1969) theorem, outer derivation as well. The basic reason for this occurrence is that every derivation $D(1)$ is not necessarily zero for P. The explicit expression for a general derivation $D$ of P turns out to be of the form

$$Df = \{a_\alpha, f\} + \beta(f - \alpha p\, \partial f/\partial p - (1-\alpha) q\, \partial f/\partial q), \qquad f \in \mathrm{P} \tag{3}$$

for some $a_\alpha \in \mathrm{P}$, $\alpha, \beta \in \mathbb{C}$.

Now two Lie algebras cannot be isomorphic if their derivation algebras are not. As a result, the Dirac quantisation prescription, Poisson bracket → operator commutation relation, cannot be uniquely effected. This occurrence has also been noted by Chernoff (1981).

Joseph has been able to identify respective subalgebras $R_i$ and $\mathscr{R}_i$ of P and $\mathscr{P}$ which have strikingly similar derivation properties. Namely ad $R_i$ and ad $\mathscr{R}_i$ form ideals of codimension 1 in $D(R_i)$ and $D(\mathscr{R}_i)$ respectively, where by D(U) we denote the derivation algebra of a Lie algebra U. Moreover he has determined that there exists an isomorphism between such pairs of subalgebras and has suggested that the solution of Dirac's problem should be sought within the framework of these isomorphisms.

We presently generalise Joseph's prescription by adopting the following viewpoint. The quantisation process involves a mapping between two suitably selected isomorphic Lie subalgebras of P and $\mathscr{P}$. Presumably, the bigger the selected subalgebras the richer the resulting quantisation scheme.

A straightforward case of isomorphic subalgebras occurs when they are both simple or semisimple, in which case they possess only inner derivations. Examples of simple subalgebras of P are the ones generated by $(p^2, q^2, pq)$, $(q, pq^2, qp)$ and $(p, q^2p, qp)$. Each of these algebras happens to be isomorphic to SL(2, $\mathbb{C}$). Clearly, none of them is rich enough to implement the quantisation of physical systems which include interactions.

Non-trivial subalgebras of P which form a basis for quantisation have been suggested, from a different viewpoint, by Bayen *et al* (1978). We are referring to their so-called good observables which form a restricted class of functions† on phase space constituting a subalgebra G of P. The latter is characterised by the fact that each of its elements generates, by the Poisson bracket, a group of symplectic diffeomorphisms in phase space. Moreover, it has been suggested by Bayen *et al* (1978) that this restricted class of phase space functions has a predominant physical content at the classical level.

It should be mentioned that the viewpoint of Bayen *et al* regarding quantisation is different from the one customarily adopted, i.e. as a mapping from phase space functions to quantum operators. The above authors see quantisation as a deformation of the Poisson product in P. One such deformation, with which we shall be dealing later on, is given by

$$\{f, g\} \to \{f, g\}_\lambda = \{f, g\} + \lambda C(f, g) \tag{4}$$

where $C$ is a suitable two-cochain in P (Goto and Grosshans 1978).

The algebra of good observables plays a determining role concerning the choice of a deformation product. We shall not discuss the details of the deformation approach to quantisation. The interested reader can look up Bayen *et al* (1978). What we do find remarkable is the fact that two algebras of good observables identified by Bayen *et al* for the Kepler two-body problem happen to be simple. This means that each of them possesses only inner derivations. Furthermore, they can be mapped on isomorphic subalgebras of operators in $\mathscr{P}$ and thus satisfy our quantisation criterion.

Encouraged by the specific realisations stemming from the Kepler two-body problem, we shall now seek a more general result. In particular we ask whether the

† We point out that in the analysis of Bayen *et al* (1978) one considers $C^\infty$ functions on phase space. In our case we shall restrict ourselves to polynomial functions, i.e. elements of P. Thus by functions on phase space we mean polynomial functions.

deformation of P given by (4) leads to an algebra $P_\lambda$ of classical functions which possess only inner derivations. If such becomes the case, then a mapping from $P_\lambda$ to $\mathcal{P}$ is conceivable.

Let $D$ be a derivation of $P_\lambda$. Following Dixmier's (1977) notation we write

$$D(p) = a_q, \qquad D(q) = -a_p \tag{5}$$

where $a_q, a_p \in P$.

Owing to the antisymmetry of $C(f, g)$ we obtain

$$\{p, p\}_\lambda = \{p, p\} + \lambda C(p, p) = 0 \tag{6a}$$

and in the same manner

$$\{q, q\}_\lambda = 0. \tag{6b}$$

From $(6a)$ and $(6b)$ we get

$$\{Dp, p\}_\lambda + \{p, Dp\}_\lambda = \{Dq, q\}_\lambda + \{q, Dq\}_\lambda = 0. \tag{7}$$

Similarly from

$$\{p, q\}_\lambda = 1 + \lambda C(p, q) \tag{8}$$

we have

$$D(1) + \lambda DC(p, q) = \{Dp, q\}_\lambda + \{p, Dq\}_\lambda = \{a_q, q\}_\lambda + \{p, -a_p\}_\lambda \tag{9}$$

or

$$D(1) + \lambda DC(p, q) = \{a_q, q\} + \lambda C(a_q, q) + \{a_p, p\} + \lambda C(p, -a_p). \tag{10}$$

But

$$\{a_q, q\} = -\partial a_q/\partial p, \qquad \{a_p, p\} = \partial a_p/\partial q \tag{11}$$

from which follows

$$D(1) + \lambda DC(p, q) = -\partial a_q/\partial p + \lambda C(a_q, q) + \partial a_p/\partial q + \lambda C(p, -a_q). \tag{12}$$

In order to formulate a sufficiency condition that $P_\lambda$ has only inner derivations we proceed as follows.

To begin with, if $D$ is inner $a_p$ and $a_q$ should be expressible in the form

$$-a_p = \{q, B\}_\lambda = \{q, B\} + \lambda C(q, B) = \partial B/\partial p + \lambda C(q, B) \tag{13}$$

for some $B(p, q) \in P$.

Similarly we should have

$$a_q = -\partial B/\partial q + \lambda C(p, B). \tag{14}$$

Take $B(p, q)$ to be of the form

$$B(p, q) = b(p, q) + b_1(q) + b_2(p). \tag{15}$$

It follows that

$$\partial B/\partial q = \partial b/\partial q + \partial b_1/\partial q = \partial b/\partial q + f(q) \tag{16}$$

and

$$\partial B/\partial p = \partial b/\partial p + \partial b_2/\partial p = \partial b/\partial p + g(p). \tag{17}$$

Substituting (13) and (14) in (16) and (17) respectively, we obtain

$$-a_p = \partial b/\partial p + g(p) + \lambda C(q, B), \tag{18}$$

$$a_q = -\partial b/\partial q - f(q) + \lambda C(p, B). \tag{19}$$

From (18) and (19) it follows that

$$\partial a_p/\partial q = -\partial^2 b/\partial q\, \partial p - \lambda\, \partial C(q, B)/\partial q, \tag{20}$$

$$\partial a_q/\partial p = \partial^2 b/\partial q\, \partial p + \lambda\, \partial C(p, B)/\partial p. \tag{21}$$

Finally, substituting in condition (12), we get

$$D(1) + \lambda DC(p, q) = \lambda C(Dp, q) + \lambda C(p, Dq) - \lambda \frac{\partial}{\partial q} C(q, B) - \lambda \frac{\partial}{\partial p} C(p, B). \tag{22}$$

This is the desirable condition. If for a given choice of $B$ there exists a cochain $C$ satisfying (22) then $P_\lambda$ has only inner derivations.

A particular choice occurs if $C(q, B)$ is a function of $p$ only and $C(p, B)$ a function of $q$ only. For such a situation condition (22) assumes the simple form

$$D(1) = \lambda[C(Dp, q) + C(p, Dq) - D(p, q)]. \tag{23}$$

One might suspect that the above condition leads to the trivial requirement that $D(1) = 0$. However, despite an extensive search of the literature, we were not able to identify a proof that $DC(p, q)$ is equal to $C(Dp, q) + C(p, Dq)$, even for derivations $D$ of P.

A more interesting possibility takes place if one sets

$$C(p, B) = pf(p, q), \qquad C(q, B) = qf(p, q). \tag{24$a, b$}$$

It follows that

$$\partial C(p, B)/\partial p = f + p\, \partial f/\partial p, \qquad \partial C(q, B)/\partial q = f + q\, \partial f/\partial q. \tag{25$a, b$}$$

Substituting in (22) we obtain

$$D(1) + \lambda[DC(p, q) - C(Dp, q) - C(p, Dq)] = -\lambda(2f + p\, \partial f/\partial p + q\, \partial f/\partial q). \tag{26}$$

The right-hand side of the above expression bears striking resemblance to the outer derivation term entering Wollenberg's theorem. One suspects that condition (26), for a suitable choice of $C$ and $D(1)$, accounts for the absence of Wollenberg's term from $P_\lambda$ derivations.

It is remarkable that the deformation product offers the possibility for constructing classical algebras $P_\lambda$ of phase space functions which have the same property as the algebra $\mathcal{P}$ of operators, namely that every derivation of $P_\lambda$ is inner.

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